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Self-Consistency of the Quantum Billiard Problem in Wormhole Spacetimes

Frank Antonsen[1] and Karsten Bormann[1]

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A wormhole can be made to function as a time machine. In the context of the quantum billiard problem in the presence of a wormhole we examine whether this is compatible with the self-consistency of physics. We derive a self-consistency condition in which the classical limit corresponds to known results for the (classical) billiard problem in a wormhole space-time and that suggests that some fine-tuning of initial conditions might be necessary.

1. INTRODUCTION

Consider a flat space-time in which two regions (mouth A and mouth B of Fig. 1) are connected by a throat (a wormhole) and assume that the intrinsic length of this wormhole is small compared to the distance between mouth A and mouth B in the external, flat space-time. It can be shown that this geometry can lead to the existence of closed timelike curves either by, at some period of time, accelerating one mouth of the wormhole relative to the other or by placing the mouths in regions of differing gravitational potentials (Echeverria *et al.*, 1991; Morris *et al.*, 1988; Kim and Thorne, 1991; Friedman *et al.*, 1990). An observer on a closed timelike curve can influence not only his own future but also his past; the wormhole thus can be made to function as a time machine. It is this time machine effect that we shall investigate in the following. We will leave out of consideration the specifics of the wormhole and *only* discuss the question of whether the existence of closed timelike curves can be accommodated within a self-consistent physics.

Now consider the *classical* billiard problem in the presence of a wormhole in (for simplicity) an otherwise flat space-time. A ball is incident upon

[1] Niels Bohr Institute, DK-2100 Copenhagen Ø, Denmark.

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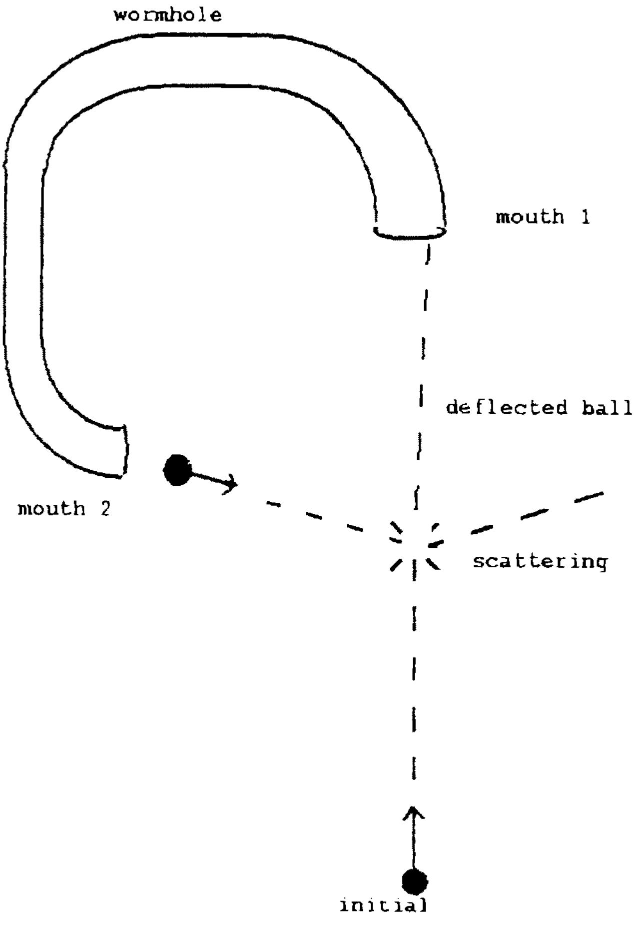

Fig. 1. The setting of the classical billiard ball problem in a wormhole space-time; the particle moves through the wormhole and backward in time, hitting itself in the past, but still reaching the mouth in accordance with the principle of self-consistency.

the one mouth, A, of the wormhole, goes through the wormhole, and thereby gets shifted backward in time giving the possibility that the ball hits itself so hard that it no longer reaches mouth A. But if it does not reach the mouth, it does not come out of mouth B and therefore does not hit itself and therefore reaches mouth A . . . (the hen and the egg problem). One then proceeds to invoke the principle of self-consistency (Echeverria *et al.*, 1991; Novikov, 1989; Morris *et al.*, 1988; Kim and Thorne, 1991; Friedman *et al.*, 1990; Lossev and Novikov, 1991); the time-shifted ball is only allowed to hit itself a little bit, such that it still reaches the hole, but on a slightly different path, which in turn is the reason that it only hits itself a little bit, thus rendering the motion self-consistent, i.e., the ball is allowed to travel on a closed timelike curve (a CTC) only if the associated motion is self-consistent.

A study of the self-consistency of the *quantum mechanical* behavior in a space-time with a wormhole is pursued using the simple model of wave packets whose size are small compared to that of the wormhole mouth. Thus we can ignore the possibility that only part of the wave packet enters the wormhole, and we can neglect scattering of the wave packet off the wormhole mouths. The wormhole itself will thus have very little effect on the wave packet, besides moving it to a different point in space-time. We assume that the wormhole is doubling as a time machine thus making it possible for the incoming wave packet to scatter on its own, time-shifted "self." As the wave packet, and thus the corresponding field, are self-interacting, self-consistency is not *a priori* fulfilled. The question of self-consistency is pursued by investigating the scattering of the incoming wave packet upon (the potential derived by) its time-shifted "self." The requirement that the physics of the system be self-consistent leads to a closed equation which the wave packets have to satisfy. Requiring these solutions to be stable further constrains the form of the *incoming* wave packet, i.e., the *initial* conditions.

This is the obvious way of generalizing this classical treatment to a quantum mechanical one, but unfortunately it is not without problems. A space-time possessing closed timelike curves is not foliable, and hence we cannot, *in the vicinity of the wormhole,* make the 3 + 1 splitting of space-time essential to Schrödinger mechanics. This problem can be overcome when, instead of including the wormhole directly, we include it only in an effective theory which is such that we do not have to use Schrödinger theory near the wormhole but only sufficiently far away from it where we empirically know ordinary quantum theory to be correct. In this effective theory, the possibility of going through the wormhole and backward in time gives rise to an interaction which looks much like an ordinary self-energy diagram. Away from the wormhole, Schrödinger mechanics must be valid, but we have to take the self-interactions introduced by the presence of a time machine into account.

The Hamiltonian chosen to parametrize the time machine function of the wormhole as mentioned is (almost) the simplest possible: One that destroys the particle/wave when it enters the (given region surrounding) one wormhole mouth and created at some earlier time, i.e., when it exits (the region surrounding) the other wormhole mouth. One cannot *a priori* assume that such a simple Hamiltonian would yield a physical theory, so in order to examine this point we argue in Section 2.1 that this effective theory describing the wormhole action is mathematically equivalent to the nonlinear Schrödinger equation.

Other approaches to the problem of creating a quantum mechanics valid in the presence of a wormhole have been suggested by various authors—their program is of course much more ambitious than ours in that they try to create a full-fledged quantum theory, while we are content with having an effective theory. It has been suggested by Klinkhammer and Thorne (n.d.) that a path-integral formulation was still possible, and this approach has been applied by several authors (Friedman *et al.*, 1992; Boulware, 1992; Politzer, 1992; Hartle, 1993). Another method has been developed by Deutsch (1991); he finds that the pathologies present in a classical treatment are absent or at least mitigated in a quantum mechanical treatment. While closed timelike curves restrict the initial data in a classical setup, he finds that this is not so when the analysis is carried out quantum mechanically. Contrary to this, we find that self-consistency imposes restrictions on the initial data in a quantum description—especially when we also impose a restriction on the *form* of the wave packets. It should be noted, though, that self-consistency equation is so complicated that we do not know to what extent it restricts the initial conditions in the general case (the case where there is no restriction on the form of the wave packet traveling on the CTC). The other authors find that a free particle theory is consistent, but that interaction leads to nonunitarity (Deutsch also finds a loss of unitarity); a functional integration approach is, however, still possible. The model of self-interacting fields in a wormhole space-time put forward in this paper also leads to nonunitarity, but our treatment of the model avoids this problem (or rather: it is hardly visible in the way we apply our formalism to the quantum billiard problem), but in light of our findings, we should probably expect that the path integration measure would have to be nontrivial in order to impose self-consistency.

It has been suggested by Kim and Thorne (1991) and Lossev and Novikov (1991) that the problems in the treatment of the classical billiard problem could be solved by using a quantum mechanical treatment. To investigate the relationship with the classical billiard problem, a Gaussian wave packet is substituted for the balls. It is shown that self-consistency is exceedingly difficult to obtain, so in this case the *principle of self-consistency* really amounts to a *fine tuning* of *initial* parameters, which basically is also, in a somewhat milder version, the classical result.

To keep things so simple that an analytical solution to the problem is possible we will assume that the only effect of the wormhole on the wave packet, besides moving it to a different place in space-time, is the possibility of a shift in momentum. The diverging lens effect of the wormhole (Kim and Thorne, 1991) will be ignored, as will the scattering of the wave upon the wormhole mouths. We emphasize that we are *not* studying the scattering of a wave packet off a wormhole mouth (considered as a perturbation of flat space-time), we are only interested in the possibility or impossibility of self-consistent motion. Therefore we think that this crude model of the wormhole's interaction with its surroundings should suffice. Any quantum mechanical model taking the wormhole (and the resulting absence of a foliation) into consideration, e.g., the path-integration approach, has to be equivalent to the Schrödinger theory sufficiently far away, where space-time is supposed to be flat. Hence sufficiently far away, any model has to be equivalent to ours, although perhaps with a different scattering kernel. If not, the mere applicability of Schrödinger quantum theory today would exclude the existence, anywhere in the universe, of regions with closed timelike curves.

In Section 2 we examine what it takes to fulfill the self-consistency requirement in the case of a nonrelativistic theory with Coulomb interactions. Thus we will have an incoming Gaussian wave at "infinity" moving toward mouth A, into it, and emerging from mouth B some time in the past, the time step being such that it scatters upon itself as it is on its way from infinity to mouth A (all of this happening in a flat space-time). To have self-consistency the scattering of the incoming wave upon its future (time-shifted) "self" should have constant amplitude if one chose to iterate the above process (as was the case in the classical billiard problem). This is the self-consistency requirement.

In Section 3 we examine what it takes for a given solution to the self-consistency equation to be stable against small perturbations. By analogy with the classical billiard problem discussed above, this is done to suggest the degree of fine tuning necessary to ensure that the balls do not hit each other so hard that the incoming ball does not reach the wormhole.

2. THE SELF-CONSISTENCY REQUIREMENT FOR A NONRELATIVISTIC FIELD WITH COULOMB INTERACTIONS IN 2+1 DIMENSIONS

From nonrelativistic quantum mechanics, we know that wave packet ψ_i can undergo a transition $\psi_i \to \psi_f$ in the presence of a perturbation. To first order, perturbation theory gives

$$\psi_f(\mathbf{k}') = \int \frac{V_{\mathbf{k}\mathbf{k}'}}{\epsilon_{\mathbf{k}'} - \epsilon_{\mathbf{k}}} \psi_i(\mathbf{k})\, d^n k \tag{1}$$

where $V_{\mathbf{k}\mathbf{k}'} = \langle \mathbf{k}' | V | \mathbf{k} \rangle$ is the matrix element of the perturbation, n denotes the number of spatial dimensions, and $\epsilon_{\mathbf{k}}$ is the energy, $\epsilon_{\mathbf{k}} = k^2/2m$. We will choose units in which $m = 1$.

The case of the quantum billiard problem is quite special: The potential in which ψ_i scatters is derived from its future "self," ψ_f [which gives rise to a charge distribution $\rho(\mathbf{x}, t) = |\psi_f(\mathbf{x}, t)|^2]$, i.e., $V = V[\psi_f]$. We will choose a potential of the form

$$V(r) = \alpha' \rho(\mathbf{x}, t) r^{\epsilon - 1} = \alpha' |\psi_f(\mathbf{x}, t))|^2 r^{\epsilon - 1} \equiv v(r) |\psi_f(\mathbf{x}, t)|^2 \tag{2}$$

where ϵ is taken to be small. One must either choose this prescription for an almost Coulomb potential or screen the Coulomb potential in order to obtain a finite theory. Some comments on the case of the screened potential (the Yukawa potential) will be made at the end of this section.

Denoting the Fourier coefficients of the original state by $a_{\mathbf{k}}$ and those of the scattered state (that which goes through the wormhole and scatters its former "self") by $c_{\mathbf{k}}$,

$$\psi_i(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}} \int a_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}}\, d^n x$$

$$\psi_f(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}} \int c_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}}\, d^n x$$

we obtain upon insertion in equation (1), by using the Fourier convolution theorem (a tilde denotes Fourier transform)

$$\widetilde{fg}(k) = (\tilde{f} * \tilde{g})(k) = \int \tilde{f}(p) \tilde{g}(k - p)\, d^n p$$

twice and changing variables a couple of times,

$$c_{\mathbf{k}'} = \alpha' \int \frac{a_{\mathbf{k}}}{\epsilon_{\mathbf{k}'} - \epsilon_{\mathbf{k}}} c_{\mathbf{l}}^* c_{\mathbf{q}-\mathbf{l}} \|\mathbf{k} - \mathbf{k}' - \mathbf{q}\|^{\epsilon - 2}\, d^n q\, d^n l\, d^n k \tag{3}$$

$$\equiv \int c_{\mathbf{p}}^* \hat{X}_{\mathbf{k}'}^{\mathbf{p},\mathbf{q}} c_{\mathbf{q}}\, d^n p\, d^n q \tag{4}$$

where we have introduced a scattering kernel

$$\hat{X}_{\mathbf{k}'}^{\mathbf{p},\mathbf{q}} \equiv \int \frac{a_{\mathbf{k}} \tilde{v}(\mathbf{l})}{\epsilon_{\mathbf{k}'} - \epsilon_{\mathbf{k}+\mathbf{l}+\mathbf{p}+\mathbf{q}}}\, d^n k\, d^n l \tag{5}$$

$$= 2 \int \frac{a_{\mathbf{k}} \tilde{v}(\mathbf{l})}{(\mathbf{k}')^2 - (\mathbf{k} + \mathbf{l} + \mathbf{p} + \mathbf{q})^2}\, d^n k\, d^n l \tag{6}$$

where $\tilde{v}(k) = \alpha' \|k\|^{\epsilon - 2}$ is the Fourier transform of $v(r) = \alpha' r^{\epsilon - 1}$ and where we have inserted $\epsilon_{\mathbf{k}} = \frac{1}{2}\mathbf{k}^2$.

The following procedure will give us the self-consistent solutions:

- Choose an initial wave function ψ_i, denote its Fourier coefficients by $a_{\mathbf{k}}$.

- Calculate the scattering kernel $\hat{X}^{\mathbf{p};\mathbf{q}}_{\mathbf{k}'}$ from $a_{\mathbf{k}}$.
- Solve the (infinite) set of quadratic equations (self-consistency relations)

$$c_{\mathbf{k}'} = \int c^*_{\mathbf{p}} \hat{X}^{\mathbf{p};\mathbf{q}}_{\mathbf{k}'} c_{\mathbf{q}} \, d^n p \, d^n q$$

In a more general setup the kernel would also contain information about the structure and geometry of the wormhole. Thus it is essentially this quantity which hides our ignorance of the detailed structure of the wormhole.

2.1. On the Consistency and Limitations of this Formalism

Now, as mentioned in the introduction, space-times with closed timelike curves do not admit a foliation, and hence ordinary quantum mechanics is in principle meaningless. Thus, some comments on the consistency of the proposed formalism are in order. First of all, we do not attempt to give a quantum description of the dynamics close to the wormhole mouths. Sufficiently far outside that region, where space-time is almost flat, foliations are possible, and ordinary quantum mechanics is known to be valid. But we are interested in interaction of this "forbidden region" with its surroundings, as we want the wave packet to traverse the wormhole. We then try to set up an *effective* theory which can accommodate this. The "forbidden region" is considered as a kind of "black box" which interacts with the environment: particles can enter it, and it emits particles, too. This would be completely analogous to the situation of a quantum mechanical system interacting with a classical system, were it not for the added feature of special (temporal) correlations. A particle entering the region at time t is correlated with a particle exiting at the earlier time $t - T$, where T is the typical time step of the wormhole. If the original wave packet is to interact with it, we can describe this in terms of an effective interaction $V(r) = v(r)|\psi_f(\mathbf{x}, t)|^2$, where $v(r)$ is the potential between two classical point particles and $|\psi_f|^2$ is the (normalized) "charge" distribution. We can reexpress the potential in terms of the initial wave packet by writing $|\psi_f(\mathbf{x}, t)|^2 = w(\mathbf{x}, t)|\psi_i(\mathbf{x}, t)|^2$, whereby the effective potential becomes $V(\mathbf{x}, t) = w(\mathbf{x}, t)|\psi_i(\mathbf{x}, t)|^2$. This holds provided we stay away from the at most countable number of zeros of ψ_i, which thus form a set of measure zero. We notice, by the way, that the potential no longer *a priori* is radial, and will in general be time dependent, too. Simply plugging this into a Schrödinger equation leads to the following effective equation of motion (dropping the subscript i on ψ):

$$-\frac{1}{2}\nabla^2\psi + w(\mathbf{x}, t)|\psi|^2\psi = i\frac{\partial}{\partial t}\psi \tag{7}$$

which is a slight generalization of the well-known nonlinear Schrödinger equation (Taniuti and Yajima, 1969), the only new feature being the nonconstant coefficient $w(\mathbf{x}, t)$.

Giving up describing the dynamics inside the "forbidden region," we can essentially use ordinary Schrödinger mechanics outside, but with an effective potential depending upon the wave function, thus leading to a generalization of the nonlinear Schrödinger equation as the effective equation of motion.[2] Hence, as an effective theory, the proposed formalism should suffice. Thus, also, in principle, one could calculate all sorts of transition amplitudes using the scattering theory of this generalization of the nonlinear Schrödinger equation.

2.2. Gaussian Wave Packets in an Almost Coulomb Potential

The simplest nontrivial spatial dimensionality is two and to make things simple we will restrict ourselves to that. We expect the results to change only slightly in $n > 2$, i.e., only small changes in numerical values and perhaps in the number of solutions are expected. The order of the Bessel function could change, and we would get a factor $(k')^{n-2}$ in the final result.

With a Gaussian wave packet as our initial wave function, parameterized as

$$a_{\mathbf{k}} = e^{-ak^2+\mathbf{b}\cdot\mathbf{k}+c} \tag{8}$$

we obtain after a lengthy and tedious but standard calculation

$$\hat{X}^{\mathbf{k}'}_{\mathbf{pq}} = 2\alpha'\pi^2 B(\epsilon, 1-\epsilon) \int_0^\infty e^{-ax^2+c'} I_0(b'x) \frac{x}{(k'^2 - x^2)^{1-\epsilon}} \, dx \tag{9}$$

where $B(x, y) \equiv \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the beta function, I_0 is a modified Bessel function, and the coefficients are

$$b' = \|\mathbf{b} - 2a(\mathbf{p} - \mathbf{q})\| \tag{10}$$

$$c' = c - a(\mathbf{p} + \mathbf{q})^2 - \mathbf{b}\cdot(\mathbf{p} + \mathbf{q}) \tag{11}$$

[2] One should note that the usual nonlinear Schrödinger equation, i.e., the equation with $w(\mathbf{x}, t) = w_0 =$ const, has a countable spectrum of soliton solutions. We would expect this slight generalization to behave similarly—one can consider it as a perturbation of the usual nonlinear Schrödinger equation—and hence to have a countable spectrum of soliton solutions. We could furthermore allow the particle to go through the time machine a countable number of times—this would alter the potential in the nonlinear Schrödinger equation to $V_n = w_n(\mathbf{x}, t)|\psi|^{2n}$, where n is the number of times it goes through; we see that the nonlinearity increases (not surprisingly) with the number of times the wormhole is traversed. This would be expected to yield a continuum of solutions which might be of use in a path-integration approach, probably allowing for a reduction of the Hilbert space of states.

Proceed from equation (6) by noting that the beta function B is singular in the limit $\epsilon \to 0$. This comes as no surprise, as this is the place where we put the infinities arising from the nature of the Coulomb field. Imagining the beta function regularized by renormalization makes it plausible to put $\epsilon \equiv 0$ inside the integral and simply treat the beta function taken at $\epsilon = 0$ as a (finite) constant. Doing this, we can evaluate the above integral, obtaining

$$\hat{X}^{\mathbf{k}'}_{\mathbf{pq}} = 2i\alpha'\pi^3 B(\epsilon, 1 - \epsilon)e^{-a(k')^2+c'}I_0(b'k') \tag{12}$$

Note from equations (4) and (11) that $c_{\mathbf{k}'}$ will always go like a Gaussian times some function, i.e.,

$$c_{\mathbf{p}} = f(\mathbf{p})e^{-\alpha p^2} \tag{13}$$

The self-consistency condition then reads (see Appendix for details)

$$\begin{aligned} c_{\mathbf{k}'} = {} & 2\pi\xi e^{-ak'^2} \sum_{nm} b_n b_m \int_0^\infty (p_+^2 - p_-^2)^{(n+m)/2} \\ & \times P_{(n+m)/2}\left(\frac{p_+^2 + p_-^2}{p_+^2 - p_-^2}\right) \cdot 2^{(n+m)/2} \\ & \times I_0(Ap_-)e^{-\alpha p_-^2 - (a+\alpha)p_+^2} p_+ p_- \, dp_+ \, dp_- \end{aligned} \tag{14}$$

where we have defined

$$A = 2^{3/4}\sqrt{a}k', \qquad \xi = 2i\alpha'\pi^3 B(\epsilon, 1 - \epsilon)e^{c} \tag{15}$$

with $\mathbf{p}_\pm = 2^{-1/2}(\mathbf{p} \pm \mathbf{q})$ and where b_n denotes the Taylor coefficients of $f(\mathbf{p}) = \sum b_p \mathbf{p}^n$. In general this integral is very difficult to carry out; we can, however, make a great simplification. To perform it we split the wave packet in two, one part containing only even powers of the momenta, i.e., $b_{2l+1} = 0$, and the other part containing only odd powers, i.e., $b_{2l} = 0$. The integrals can then be performed in each case (see Appendix for details) and we arrive at

$$\begin{aligned} c_{\mathbf{k}'} &= e^{-\alpha\mathbf{k}'^2} \sum_n (b_{2n}\mathbf{k}'^{2n} + b_{2n+1}\mathbf{k}'^{2n+1}) \\ &= 2\pi^2\xi e^{-ak'^2} \sum_{mn} \left[b_{2n}b_{2m}2^{n+m} \sum_{kl} \binom{n}{k}\binom{m}{l} 2^{k+l} \frac{(k+l-1)!!}{(k+l)!!} \right. \\ &\quad \times \sum_{l'} \binom{n+m-k-l}{l'} \frac{\Gamma((n+m+2-l')/2)}{\alpha^{(n+m+2-l')/2}} \\ &\quad \times \Phi\left(\frac{n+m+2-l'}{2}, 1; \frac{A^2}{4\alpha}\right) C_{k+l+l'+1} \end{aligned}$$

$$+ 2b_{2n+1}b_{2m+1}2^{n+m+1} \sum_{kl} \binom{n}{k}\binom{m}{l} 2^{k+l} \frac{(k+l-1)!!}{(k+l)!!}$$

$$\times \sum_{l'} \binom{n+m-k-l+1}{l'} \frac{\Gamma((n+m+3-l')/2)}{\alpha^{(n+m+3-l')/2}}$$

$$\times \Phi\left(\frac{n+m+3-l'}{2}, 1; \frac{A^2}{4\alpha}\right) C_{k+l+l'+1} \Bigg] \tag{16}$$

where $\Phi(a, b; z)$ is a degenerate hypergeometric function (Gradshteyn and Ryzhik, 1980)

$$\Phi(a, b; z) \equiv \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!}$$

with $(a)_n \equiv a(a+1)(a+2)\dots(a+n-1)$, and where we have introduced

$$C_\nu \equiv \begin{cases} \dfrac{(2\lambda - 1)!!}{2(2(a+\alpha))^\lambda} \left(\dfrac{\pi}{a+\alpha}\right)^{1/2}, & \nu = 2\lambda \\ \dfrac{\lambda!}{2(a+\alpha)^{\lambda+1}} & \nu = 2\lambda + 1 \end{cases} \tag{17}$$

We should notice that k' appears only through A^2 as an argument of the degenerate hypergeometric function; it thus always appears raised to an even power. From this we conclude that only wave packets with $b_{2n+1} = 0$ for $n = 0, 1, 2, \dots$ can satisfy the self-consistency requirement. Note that this is an exact result; no approximations have been used.

Equation (15) constitutes the *final form of the self-consistency requirement* in the case of an incoming Gaussian wave packet possessing a Coulomb potential, and so it is this equation we have to solve to find self-consistent solutions. In general the self-consistency requirement can only be solved (in principle) in the two extreme cases f = const and f not a polynomial (i.e., the Taylor series never terminates, in which case f would be some analytic function of k'^2); due to the hypergeometric function on the right-hand side, the self-consistency equation has no solutions when f is a polynomial—its expansion will never terminate for the values of its arguments which appear in the self-consistency requirement. As an example of the case where f is an analytical function, in the next section we consider the case where f is a pure Gaussian.

Figure 2 shows the kernel as a function of $p_\pm$ for fixed k'. Note the very smooth behavior of this function; this is what makes an analytical

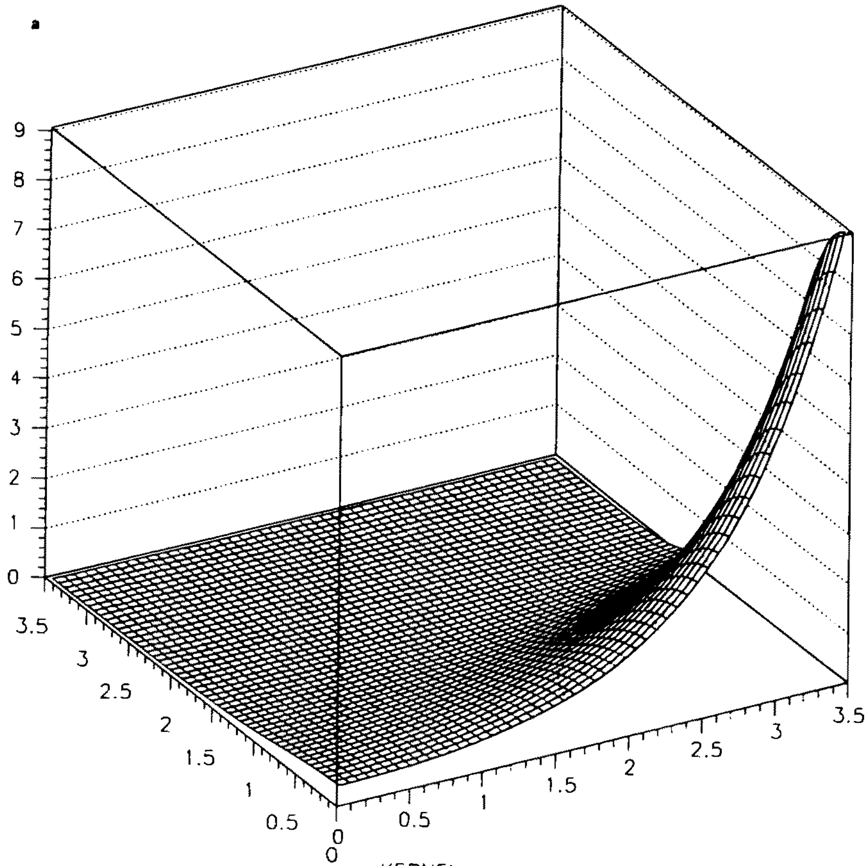

Fig. 2. The scattering kernel for an incoming Gaussian in a power-law potential. (a) Surface plot of $\hat{X}_{\mathbf{k}}^{\mathbf{p};\mathbf{q}}$ as a function of $x = \|\mathbf{p} - \mathbf{q}\|$ and $y = \|\mathbf{p} + \mathbf{q}\|$ for fixed k' ($a = 1$, $k' = 1$). (b) Contour plot representation of (a).

solution possible. Note also the very large range in which it takes its values—this makes numerical simulations impractical.

2.3. Pure Gaussian

If the wave packet is a pure Gaussian, we get a requirement on the wave packet traversing the wormhole (remember that the α refers to ψ_f, whereas the a refers to ψ_i). By putting $c_{\mathbf{k}'} = b_0 \exp(-\alpha k'^2)$ [cf. (12)] on the left-hand side and similarly on the right-hand side, where only one term in the sum would then appear, one immediately sees that

$$b_0 = b_0^2 2\pi^2 \xi \tag{18}$$

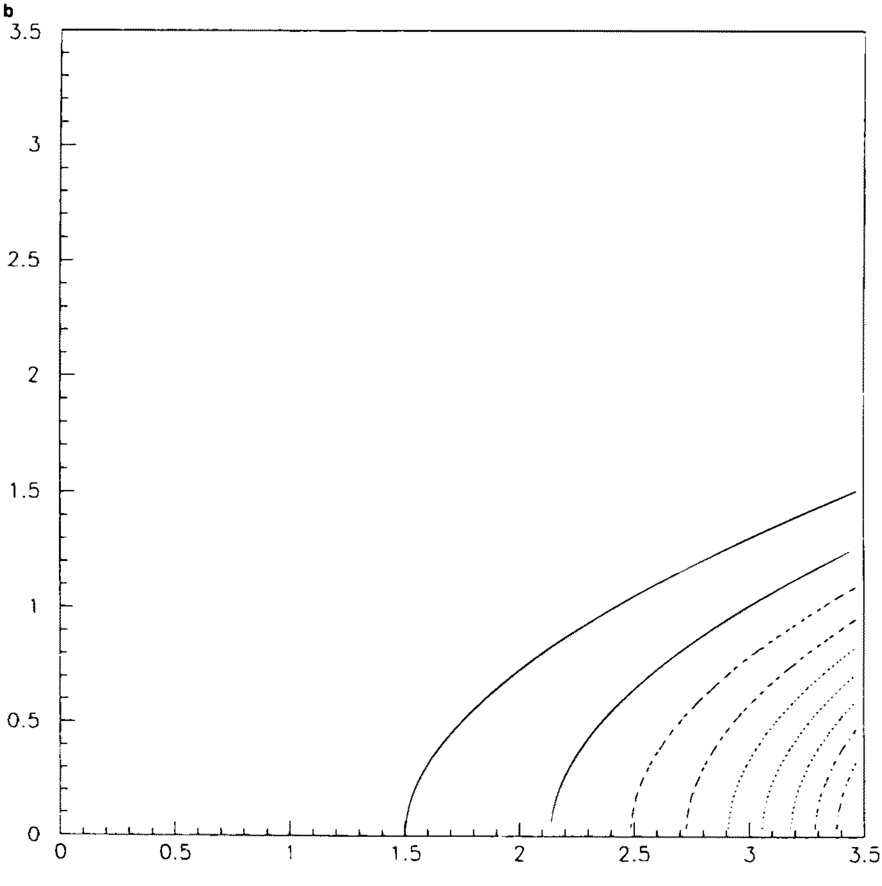

Fig. 2. Continued.

i.e., $b_0 = (2\pi^2\xi)^{-1}$ (or $b_0 = 0$, but this would correspond to $\psi \equiv 0$ and is hence not interesting). Also, by differentiating twice with respect to k' and putting $k' = 0$, one gets

$$\alpha = \frac{1}{2}a \pm \frac{1}{2}(a^2 + 2\sqrt{2}a)^{1/2} \tag{19}$$

i.e., there are exactly two solutions for the scattered wave packet.

This case, where the wave packet is Gaussian at all times (i.e., both before and after scattering), is the quantum analogue of the classical billiard problem (the Gaussian wave packets which are substituted for the billiard balls can be as localized as the Heisenberg uncertainty principle allows).

In the case where the wave packet is required to be normalized, we furthermore get a requirement on the *original* wave packet (i.e., on a). But the number of requirements grows as two times the number of terms in the Taylor expansion of $f(\mathbf{p})$, making this analysis feasible only in the case of a pure Gaussian. Normalization of the Gaussian wave packet would require $b_0 = (2\alpha/\pi)^{1/2}$, i.e., in order to have normalized wave packets we would have to impose the requirement

$$\alpha = (8\pi^3\xi^2)^{-1} = -(64ia\pi^7\alpha'^2B^2(\epsilon, 1-\epsilon))^{-1} \tag{20}$$

where we have inserted the definition of ξ [see equation (14); also, remember that α' is the coupling constant] and demanded that the original wave packet is normalized [i.e., $e^c = (2a/\pi)^{1/2}$]. This then allows only one *original Gaussian*, namely that with a satisfying[3]

$$a^2 \pm a(a^2 + 2\sqrt{2}a)^{1/2} = -(32i\pi^7\alpha'^2B^2(\epsilon, 1-\epsilon))^{-1} \tag{21}$$

Thus there are only two solutions; in units where the right-hand side is equal to one, we find $a = 0.5337543$ when we use the plus sign and $a = 1.4799995$ when we use the minus sign.

Had we chosen a Yukawa potential, $V(r) = \alpha' r^{-1} e^{-r/m}$, instead we would have had to make the substitution $k'^2 \rightarrow k'^2 + m^2$ in all the expressions, and the coefficients would become nonsingular in the limit $\epsilon \rightarrow 0$. Hence (11) would become

$$\hat{X}^{\mathbf{k}'}_{\mathbf{pq}} \propto e^{-a(k'^2+m^2)+c'} I_0(b'(k'^2 + m^2)^{1/2})$$

and the quantity A defined in (14) would become $A = 2^{3/4}\sqrt{a}(k'^2 + m^2)^{1/2}$. This would make solutions in the general case even more difficult, as the right-hand side of the self-consistency requirement (15) now would contain terms of the form $(k'^2 + m^2)^{n/2}$, where n is just some integer. This in turn would make it useless to try the expansion

$$c_{\mathbf{p}} = e^{-\alpha(p^2+m^2)} \sum_{n=0}^{\infty} b_n \mathbf{p}^n$$

The Gaussian solution would still exist, though, but with b_0 multiplied by $\exp(am^2)$. Similarly, the right-hand side of (20) would also change, but can still be taken to unity by an appropriate choice of m and α'.

[3]The factor i in equation (20) need not necessarily imply that a is complex, since the beta function $B(\epsilon, 1-\epsilon)$ strictly speaking diverges as $\epsilon \rightarrow 0$ (this was the reason for not using a proper Coulomb interaction in the first place), and hence needs regularization. In this process it could most likely take on an imaginary value (it would be forced to be nonpositive at least). We simply *assume* that we can take the right-hand side of (20) to be unity.

2.4. Stability of Solutions

In the preceding section we found solutions to the self-consistency requirement for pure Gaussians. By analogy with the classical billiard problem, where one attempts to avoid the situation in which the scattering of the incoming ball on its future "self" makes the incoming ball fail to traverse the wormhole, we now want to investigate the stability of these solutions under small perturbations. Write the self-consistency condition in symbolic form as

$$c_{\mathbf{k}'} = \hat{X}_{\mathbf{k}'}^{\mathbf{p};\mathbf{q}} c_{\mathbf{p}} c_{\mathbf{q}} \tag{22}$$

invoking a generalized summation convention consisting in integrating over repeated indices. We then consider a slight perturbation $\delta c_{\mathbf{k}}$ of a fixed solution $\zeta_{\mathbf{k}}$; to first order in the perturbation we then get

$$\delta c_{\mathbf{k}'} = \hat{X}_{\mathbf{k}'}^{\mathbf{p};\mathbf{q}} \zeta_{\mathbf{p}} \delta c_{\mathbf{q}} + \hat{X}_{\mathbf{k}'}^{\mathbf{p};\mathbf{q}} \zeta_{\mathbf{q}} \delta c_{\mathbf{p}} \equiv \hat{M}_{\mathbf{k}'}^{\mathbf{l}} \delta c_{\mathbf{l}} \tag{23}$$

where

$$\hat{M}_{\mathbf{k}'}^{\mathbf{l}} \equiv \hat{X}_{\mathbf{k}'}^{\mathbf{p};\mathbf{q}} (\zeta_{\mathbf{p}} \delta_{\mathbf{q}}^{\mathbf{l}} + \zeta_{\mathbf{q}} \delta_{\mathbf{p}}^{\mathbf{l}}) \tag{24}$$

with $\delta_{\mathbf{k}}^{\mathbf{l}} \equiv \delta^{(n)}(\mathbf{k} - \mathbf{l})$.

To study the stability of the solution $\zeta_{\mathbf{p}}$ we must consider the infinite-dimensional nonlinear map

$$\delta c_{\mathbf{k}'}^{(n)} \mapsto \delta c_{\mathbf{k}'}^{(n+1)} = \hat{M}_{\mathbf{k}'}^{\mathbf{l}} \delta c_{\mathbf{l}}^{(n)} \tag{25}$$

The difference, $\Delta_{\mathbf{k}'}^{(n)}$, between two such iterates is then simply

$$\Delta_{\mathbf{k}'}^{(n)} = |(\hat{M}_{\mathbf{k}'}^{\mathbf{l}} - \delta_{\mathbf{k}'}^{\mathbf{l}}) \delta c_{\mathbf{l}}^{(n)}| = |(\hat{M}_{\mathbf{k}'}^{\mathbf{l}} - \delta_{\mathbf{k}'}^{\mathbf{l}})^n \delta c_{\mathbf{l}}^{(0)}| \tag{26}$$

This difference then goes like the size of the eigenvalue of $\hat{M}_{\mathbf{k}'}^{\mathbf{l}}$ corresponding to $\mathbf{k}'$. Denoting this eigenvalue by $\lambda(\mathbf{k}')$, we get

$$\Delta_{\mathbf{k}'}^{(n)} \sim |(\lambda(\mathbf{k}') - 1)^n| \cdot |\delta c_{\mathbf{k}'}^{(0)}| \tag{27}$$

i.e., it diverges when $\lambda(\mathbf{k}') > 2$, in which case, then, the solution is unstable against slight perturbations. If, on the other hand, $\lambda(\mathbf{k}') < 2$, then the corresponding solution $\zeta_{\mathbf{k}}$ is stable. In accordance with the language of chaos theory, we call $\lambda(\mathbf{k}')$ the *generalized Lyapunov exponent.* In the usual finite-dimensional case treated in chaos theory, this exponential is a function of the solution ζ, which in our infinite-dimensional analogue means that λ is a functional of $\zeta_{\mathbf{k}}$.

We have reduced the problem of stability to that of finding the eigenvalues of the integral operator $\hat{M}_{\mathbf{k}}^{\mathbf{l}}$.

2.5. Stability of the Pure Gaussian

Now we examine the pure Gaussian solution found in the previous section,

$$\zeta_{\mathbf{k}} = b_0 e^{-\alpha k^2} \qquad (= c_{\mathbf{k}}) \tag{28}$$

Inserting this into the definition of the operator $\hat{M}^{\mathbf{l}}_{\mathbf{k}}$, we get, by construction, essentially the same integrals as those we performed in order to solve the self-consistency requirement. Explicitly,

$$\lambda(\mathbf{k}) = \left(b_0 e^{-ak^2}\left(\int I_0(k\|\mathbf{p} - \mathbf{l}\|)e^{-a(\mathbf{p}+\mathbf{l})^2-\alpha p^2} d^2p + (\mathbf{p} \to \mathbf{q})\right)\right)\delta^{\mathbf{l}}_{\mathbf{k}} \tag{29}$$

We can get, by ignoring the terms linear in $\mathbf{p}$, $\mathbf{q}$ in the exponent, an approximate expression for these integrals,

$$\lambda(\mathbf{k}) \approx 2b \exp\left[-\left(2a + \frac{1}{8(a + \alpha)^2}\right)k^2\right]\left(\frac{\pi}{a + \alpha}\right)^{1/2} I_0\left(\frac{k^2}{8(a + \alpha)^2}\right) \tag{30}$$

Inserting $b_0 = (\pi/\alpha)^{1/2}$ and $\alpha = \frac{1}{2}[a + (a^2 + 2a\sqrt{2})^{1/2}]$, we can plot this as a function of a and k. This is done in Fig. 3. We note the existence of a stable ($\lambda < 2$) as well as an unstable ($\lambda > 2$) region. We note that the solution is unstable for small a, but gets more and more stable as a grows, i.e., as the *original* wave packet becomes more and more localized in momentum space. But this implies that the wave packets are very diffuse in position space, i.e., do not at all look like a classical point particle. In fact, the more the quantum nature is apparent, i.e., the larger the uncertainty in position, the better the stability of the Gaussian solution. We conclude that, in accordance with intuition, the self-consistent solutions of the classical billiard problem of, e.g., Novikov (1989) in this quantum mechanical framework (localized wave packets) are unstable.

3. PLANE WAVE SOLUTIONS IN A YUKAWA POTENTIAL

As one further illustration of the method, we consider the case where the incoming wave is a plane wave

$$\psi_i(\mathbf{x}, t) = Ne^{-i\mathbf{k}_0 \cdot \mathbf{x}}$$

Furthermore, we choose to deal with Yukawa interactions represented by the potential

$$V(r) = \frac{\alpha'}{r} e^{-\mu r} \tag{31}$$

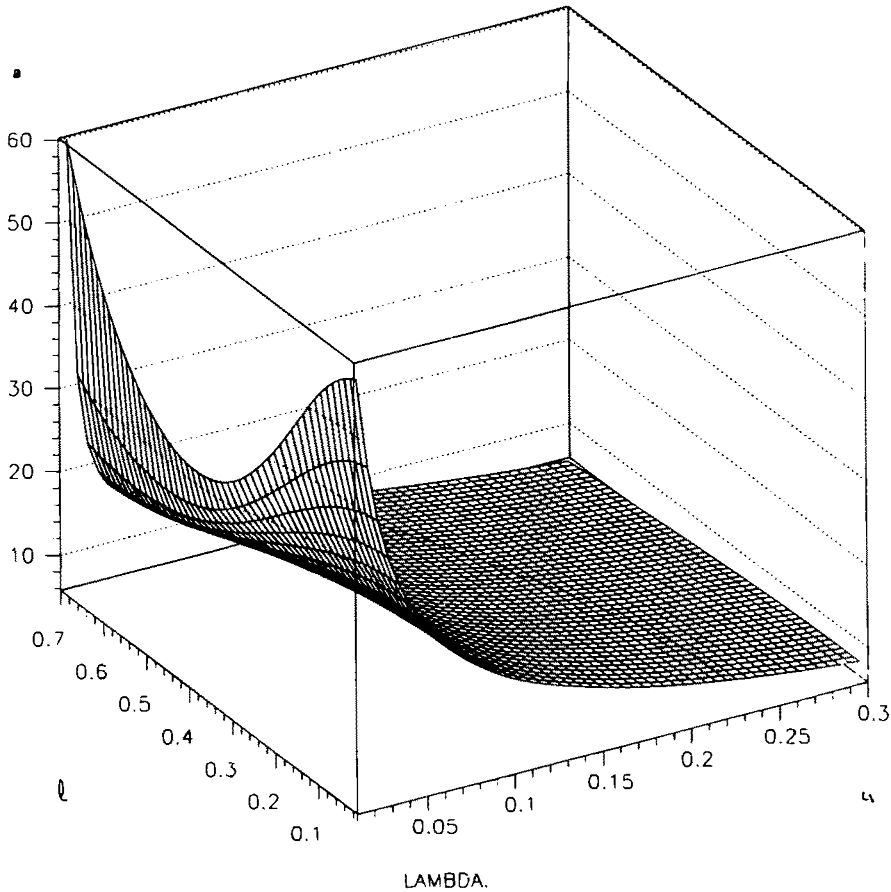

Fig. 3. The generalized Lyapunov exponent as a functional of the Gaussian solution, i.e., of a and of the momentum $\|\mathbf{k}'\|$. (a) For normed states $c_{\mathbf{k}'}$ and (b) for unnormed states.

Now we proceed by going through the same steps as in the beginning of Section 2 [equations (1)–(6)]. The Fourier coefficient $a_{\mathbf{k}}$ then becomes proportional to a delta function, $a_{\mathbf{k}} \propto \delta(\mathbf{k} - \mathbf{k}_0)$, and the integrals simplify immensely.[4] We obtain for the scattering kernel

$$\hat{X}_{\mathbf{k}'}^{\mathbf{p},\mathbf{q}} \propto \ln\left|\frac{\mu^2 - (k')^2 + \|\mathbf{k}_0 + \mathbf{p} + \mathbf{q}\|(2\mu - \|\mathbf{k}_0 + \mathbf{p} + \mathbf{q}\|)}{\mu^2 - (\mathbf{k}')^2 - \|\mathbf{k}_0 + \mathbf{p} + \mathbf{q}\|(2\mu + \|\mathbf{k}_0 + \mathbf{p} + \mathbf{q}\|)}\right| \tag{32}$$

This kernel is plotted as a function of k'/μ and $\|\mathbf{k}_0 + \mathbf{p} + \mathbf{q}\|/\mu$ in Fig. 4. Note the rather wild behavior of the function, which in turn makes it impossible to

[4]The integrals diverge for any unscreened power law potential. Hence the only regularized version of a Coulomb potential which makes the integrals converge will have to include the convergence factor $\exp(-\mu r)$. This is in contrast with the Gaussian case treated earlier, where power-law potentials were quite sufficient except in the case $\epsilon = 0$.

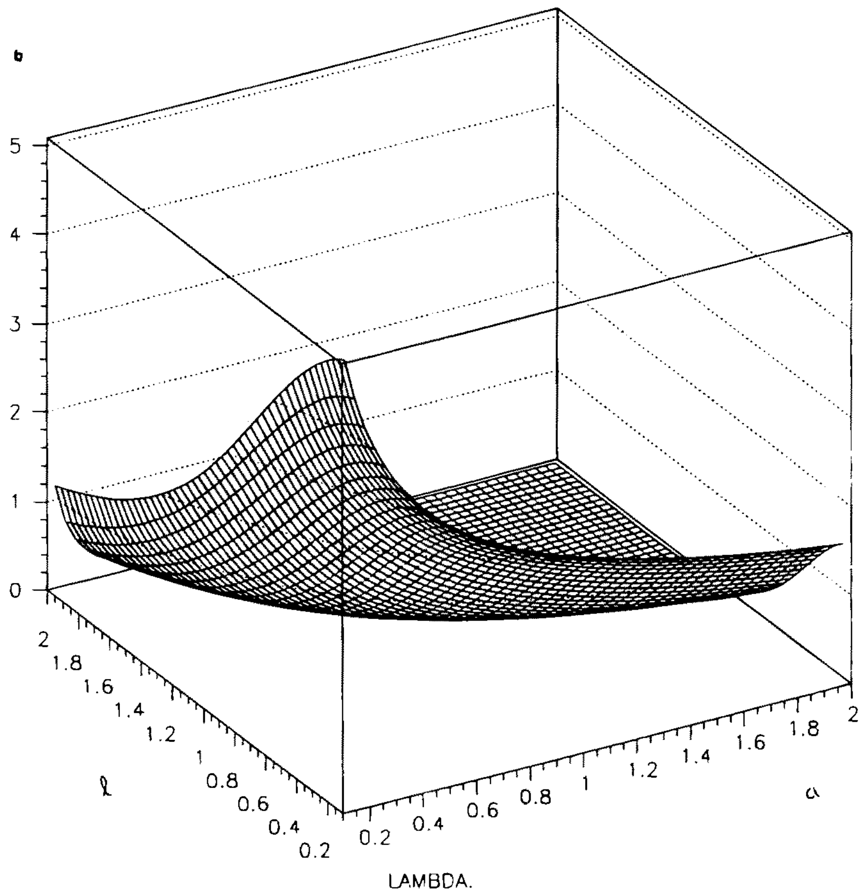

Fig. 3. Continued.

solve the self-consistency condition by numerical methods. We have not been able to find an analytical solution either.

4. DISCUSSION AND CONCLUSION

The first thing to do is to inspect some of the simplifications and assumptions made in order to enable us to find an exact solution to the problem.

One should first of all notice that we did not use any knowledge of the wormhole, nor did we specify where the interactions take place; this just has to be sufficiently (depending on the size of the wormhole) far away from it, where space-time is flat. Only the momenta of the packets were specified,

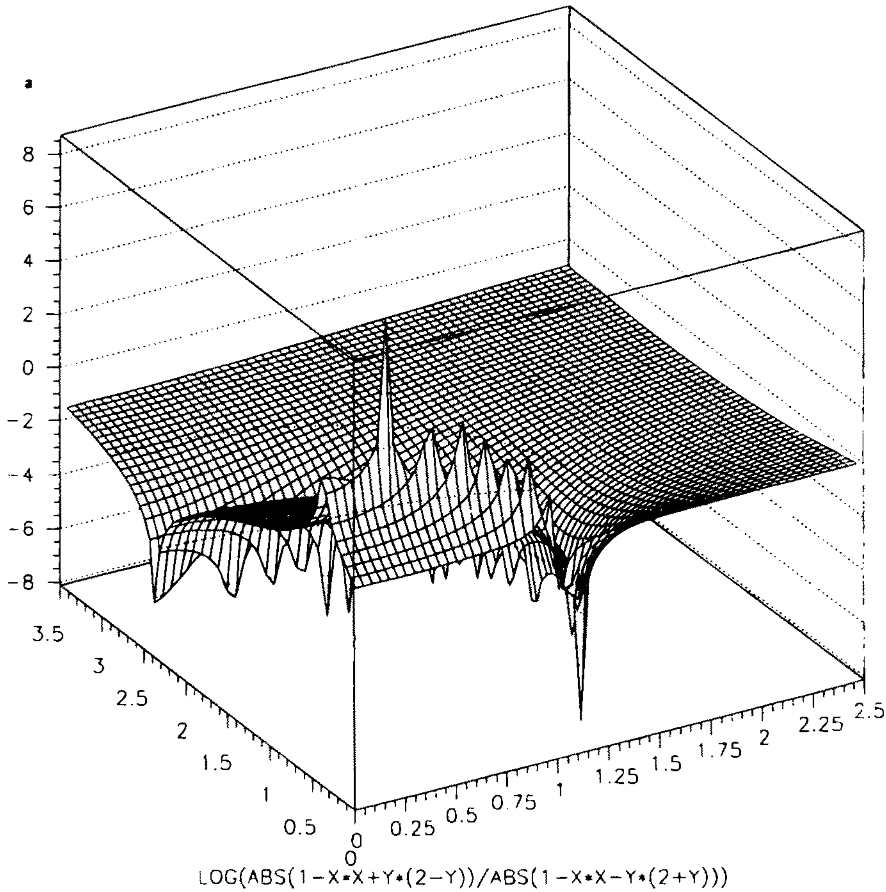

Fig. 4. The scattering kernel $\hat{X}^{p;q}_{k'}$ for the case where the incoming state ψ_i is a plane wave and the potential is a Yukawa one. (a) Surface plot of the kernel as function of $X = k'/\mu$ and $Y = \|\mathbf{k}_0 + \mathbf{p} + \mathbf{q}\|/\mu$ for fixed value of $\mathbf{k}_0$. (b) Contour plot representation of (a). k_0 is the wave number of the incoming plane wave and μ is the mass of the boson exchanged during the Yukawa interaction $V(r) \sim r^{-1}\exp(-\mu r)$.

and hence their location is indeterminate—the wormhole simply effectively introduces a new (self) interaction.

We ignored any effect the traversal of the wormhole might have on the wave packet except for a possible shift in momentum. In particular, we left the diverging-lens effect (Kim and Thorne, 1991) out of consideration as well as the scattering of the wave upon the mouths. This does not seem to us to alter the conclusions of this paper, because the resulting smaller amplitude, and consequently smaller scattering, could be compensated by changing the geometry of the problem, i.e., by changing the distance between the 'out-b

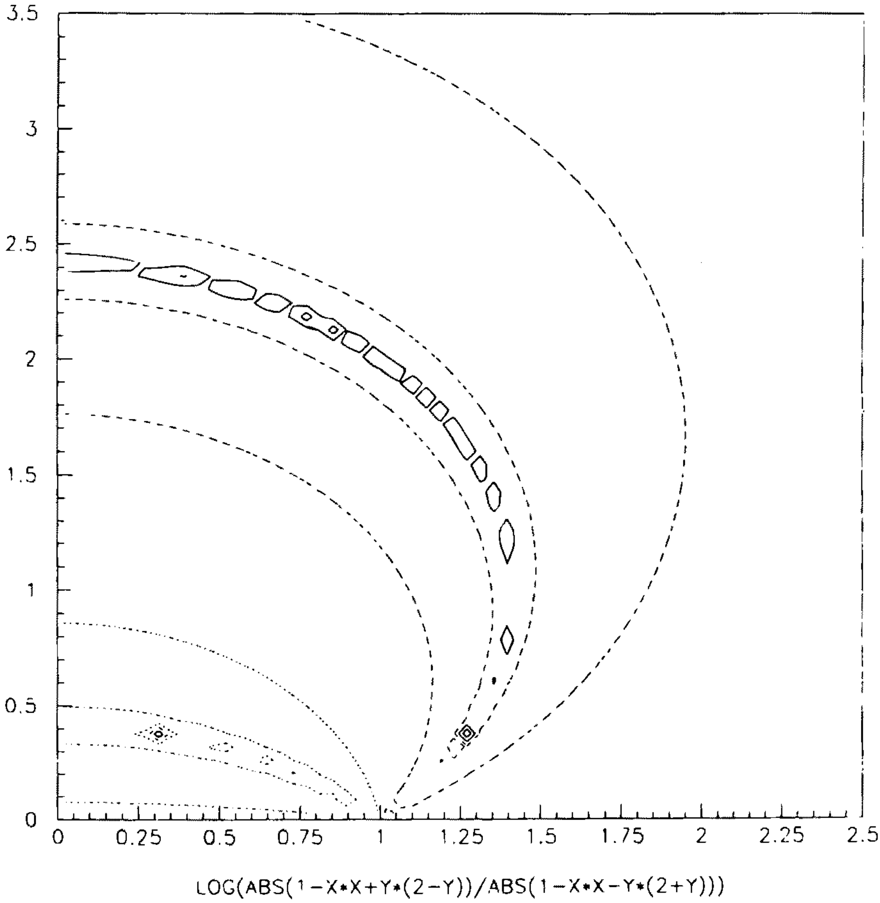

Fig. 4. Continued.

mouth' and the region where scattering occurs. Also, of course, the assumption that the wave packets are small as compared to the wormhole is essential to the calculations.

We have also ignored the possibility of the wave packet going through the wormhole more than once and therefore getting a larger shift in time (but with smaller amplitude of the shifted wave, due to the diverging lens effect), because this would just alter the region in which the scattering occurs, an effect which could be compensated again by changing the geometry appropriately. By the same token, we left out of consideration the possibility of writing the wave after scattering as a superposition of waves that have traversed the wormhole a different number of times. This problem probably could be treated in a second-quantized version of the above model, but could also be seen as just going to higher orders in the perturbation expansion and probably would not change much—it should be equivalent to a proper path-integration approach with a suitable highly nontrivial measure, taking only self-consistent solutions into account.

Thus we almost completely ignored the wormhole, which was why we could use a Hamiltonian formulation. We have only included the time-machine effect of the wormhole, and this in a rather indirect manner, through an effective potential and hence through an effective equation of motion. This equation of motion turned out to be essentially the nonlinear Schrödinger equation.

We derived a general, closed equation expressing the requirement of self-consistency. This equation could be solved exactly only in the case where the Fourier coefficients of the wave packet after scattering, i.e., the part of the wave packet traveling on the closed timelike curve, has the form $c_{\mathbf{k}'} = b_0 \exp(-\alpha k'^2)$. If the solution was normalized, we found only one possible value of the width of the incoming wave packet and that the corresponding solution was unstable in large parts of parameter space, so only fine tuning of initial conditions could render the physics self-consistent in these parts of parameter space. This need for fine tuning springs from the restrictions on the form of the wave packet. If one threw away this requirement, the need for fine tuning, i.e., the restrictions imposed upon the initial conditions, would in all likelihood become very much less severe. On the other hand, this form requirement was essential for a semiclassical picture of "billiard balls" self-interacting due to the presence of a time machine.

We were not able to find the (density of) solutions (in parameter space) in the general case where the wave packet traveling on the closed timelike curve has the form $c_{\mathbf{k}'} = f(k') \exp(-\alpha k'^2)$, but intuitively it seems that if not already constrained, one could press the need for some fine tuning by considering the case where the wave packet is large compared with the hole or the case of strong coupling because the form of the wave packet would then be more drastically disrupted. This latter remark also applies to the case where scattering of the wave packet on the (perturbation of space-time consisting of the) 'in-mouth' is included.

APPENDIX

Here we give some technical details about the calculations in the first section. The kernel is

$$\hat{X}^{\mathbf{k}'}_{\mathbf{pq}} = 2i\alpha'\pi^3 B(\epsilon, 1 - \epsilon) e^{-a(k')^2 + c'} I_0(b'k')$$

Introduce

$$\mathbf{p}_\pm = \frac{\mathbf{p} \pm \mathbf{q}}{\sqrt{2}} \tag{A1}$$

and note that we can always take $\mathbf{b} = 0$ in the original wave packet by a suitable choice of coordinates. The expression for the scattering kernel $\hat{X}^{\mathbf{k}'}_{\mathbf{pq}}$ above then splits into a Gaussian of p_+ and a modified Bessel function of p_-. The integrations over $d^2p\ d^2q$ simplify if we perform the rotation onto $\mathbf{p}_\pm$ instead. Having done that, we Taylor expand the wave packet

$$f(\mathbf{p}) = \sum_{n=0}^{\infty} b_n \mathbf{p}^n$$

where by $\mathbf{p}^n$ we mean

$$\mathbf{p}^n = (p_+^2 + p_-^2 + 2p_+p_- \cos\theta)^{n/2} \tag{A2}$$

We change to polar coordinates and note that only the angle θ between p_+ and p_- appears. Thus we carry out the integration over the angle θ_- and put $\theta_+ = \theta$, where $\theta_\pm$ is defined by $d^2p_\pm = p_\pm dp_\pm d\theta_\pm$, i.e., we make the substitution

$$d^2p\ d^2q = d^2p_+\ d^2p_- = 2\pi p_+ p_-\ dp_+\ dp_-\ d\theta \tag{A3}$$

The angular integration can be carried out using (Gradshteyn and Ryzhik, 1980)

$$\int_0^{2\pi} (a + b\cos\theta)^{k/2} d\theta = 2\pi(a^2 - b^2)^{k/4} P_{k/2}\left(\frac{a}{(a^2 - b^2)^{1/2}}\right) \tag{A4}$$

where $a = p_+^2 + p_-^2$ and $b = 2p_+p_-$ and $P_{k/2}$ is a Legendre polynomial. This gives equation (13).

With only even powers (or only odd powers) of the momenta appearing in equation (13), we can use (Gradshteyn and Ryzhik, 1980)

$$\int_0^\infty e^{-ax^2} I_0(bx) x^n\ dx = \frac{\Gamma((n+1)/2)}{2a^{(n+1)/2}}\ \Phi\left(\frac{n+1}{2}, 1; \frac{b^2}{4a}\right)$$

$$\int_0^\infty e^{-ax^2} x^n\ dx = \begin{cases} \dfrac{(n-1)!!}{2(2a)^{n/2}} \left(\dfrac{\pi}{a}\right)^{1/2} & n \text{ even} \\[2ex] \dfrac{((n-1)/2)!}{2a^{(n+1)/2}} & n \text{ odd} \end{cases}$$

$$\int_0^{2\pi} \cos^k\theta\ d\theta = 2(1 + (-1)^k)\pi \frac{(k-1)!!}{k!!}$$

to simplify equation (13), giving the self-consistency requirement the form of equation (15). Mixed terms like $\mathbf{p}^{2n}\mathbf{q}^{2m+1}$ will give a vanishing contribution due to their parity; the kernel is clearly invariant under reflections in momentum space (it only depends on the length of various combinations of $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{k}'$) and hence has even parity, but mixed terms like $\mathbf{p}^{2n}\mathbf{q}^{2m+1}$ have odd parity; thus the integrand $b_{2n}b_{2m+1}\mathbf{p}^{2n}\mathbf{q}^{2m+1}\hat{X}_{\mathbf{k}}^{\mathbf{p};\mathbf{q}}$ has odd parity. The domain over which we integrate is symmetric—it is just flat, Euclidean momentum space—whereby the integral of this mixed term vanishes. This means that also the right-hand side of the self-consistency requirement (15) splits into two sums, one containing only $b_{2n}b_{2m}$ and the other only the combination $b_{2n+1}b_{2m+1}$.

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